INVESTIGATIONS IN PHYSICAL ASTRONOMY,

PRINCIPALLY RELATIVE TO

THE MEAN MOTION OF THE LUNAR PERIGEE.

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THE difficulties that occur relative to the investigation of the mean motion of the apsids of the lunar orbit are well known. Two circumstances have principally led me to offer to the Academy the following paper on this subject. It affords me an opportunity of introducing a peculiar method of integration which I hope soon to illustrate more fully elsewhere. This method is I think deserving of the attention of mathematicians, and as applied to the integration of the equation of the lunar orbit affords very convenient results. The other circumstance alluded to, was a desire to improve a particular step that appeared defective in the lunar theory as given by several authors, and recently by M. Laplace. These circumstances and some deductions that offer themselves in the following investigation will be best understood by a brief detail relative to this subject.

The computation of the mean motion of the lunar apsids is by far the most important point in which the Newtonian Theory of the Moon, as given in the Principia, appears defective. Newton himself, in the latter editions of the Principia, seems to have abandoned the attempt to reconcile or rather deduce from Theory the motion given by Observation.

In the first edition he had made the attempt, after stating his results (Scholium, p. 462.) ; he adds, "computationes autem, ut nimis "perplexas & approximationibus impeditas, neque satis accuratas, "apponere non lubet."

It may be presumed that he found his method on examination inaccurate, otherwise it cannot be doubted he would have noticed it in the subsequent editions, and given, if not the method in detail, the results.

Machin appears the first after Newton who attempted this problem ; the inadequacy of his solution, and of those of some others of the same nature, will be noticed further on.

Clairaut, in 1748, had the honour of giving the first exact solution according to the principles of the Newtonian Theory of Gravity, after he had, in the Memoirs of the Royal Academy of Sciences at Paris, announced that the Newtonian law was inexact, inasmuch that the mean motion of the lunar apsids deduced from that law did not agree with observation.

Clairaut's result was confirmed by Euler, D'Alembert and Mayer, and subsequently by other mathematicians. Their researches, however, being directed more towards a general theory of the lunar motions, than towards the particular question of the mean motion of the apsids, are so complicated, that the exact thread of reasoning respecting this motion cannot without difficulty be

traced. The integration of the principal equation is usually commenced by supposing a formula which depends on the knowledge of that integration or on the result of observations. Thus in the Theory of the Moon by M. Laplace, he has, * u being the reciprocal of the Moon's Distance from the Earth,

$$u = \frac{1}{h^2(1+\gamma^2)} \left\{ 1 + \tfrac{1}{4}\gamma^2 + e \cos(v - \pi) \text{ \&c.} \right\}$$

and remarks

" Cette valeur de u suppose l'ellipse lunaire immobile ; mais on verra bientot qu'en vertu de l'action du soleil, les nœuds & le perigee de cette ellipse sont en mouvement. Alors, en designant par $(1-c)\,v$ le mouvement direct du perigee, &c.

$$u = \frac{1}{h^2(1+\gamma^2)} \left\{ 1 + \tfrac{1}{4}\gamma^2 + e \cos(cv - \pi) \text{ \&c.} \right\}$$

This is his first approximation, but certainly the first approximation should be the former, and the second should be deduced therefrom by a regular process. The result undoubtedly confirms this hypothesis, but it seems more consonant to the usual steps of mathematical reasoning to deduce one from the other, this is an object in the following investigation, in which also the mean motion of the perigee is computed by confining the process principally to this point, and therefore will be easily intelligible to those who may be unwilling to encounter the formidable calculations necessary for the complete lunar theory.

The method of integration which is applied to the differential equation of the lunar orbit is peculiarly convenient for the above purposes. With respect to this method it does not seem necessary here to remark more respecting it, than only to mention, that it principally derives its convenience from certain theo-

* Mec. cel. p. 187. Liv. 7. Tom. 3.

rems for finding Fluxions per Saltum. The method will be easily understood without the theorems. These and important applications thereof it is intended soon to give in a separate volume.

In order to simplify the computation as much as possible, the plane of the lunar orbit has been supposed coincident with the plane of the ecliptic, the orbit of the earth without excentricity, and the approximation has been only carried to the first power of the excentricity of the lunar orbit. These circumstances have little effect on the quantity of the mean motion of the Lunar Perigee.

The quantity of the motion found is expressed in terms of the quotient of the periodic time of the moon by the periodic time of the earth, and thereby are satisfactorily shewn the erroneous conclusions of Machin, Walmsley, Frisi and Matthew Stewart, who imagined that the mean motion of the apsids could be investigated by considering the moon acted on only by a centripetal force, the mean tangential force being = o. This is of some importance, as authors have recently referred to these solutions as exact. Professor Playfair, indeed, in his outlines of Natural Philosophy, published in 1814, speaks (vol. 2. p. 261) with some doubt on the subject. After giving Dr. Stewart's result, and referring to those of the others, he says, " The result of these investigations, " therefore, agrees nearly with observation, but it cannot be " denied that the principle on which they are founded is liable " to some objections, so that if it were not for the information " derived from the direct solution of the problem of the three " bodies, it might still be doubted, whether the principle of gravity " accounted exactly for the motion of the Moon's apsids."

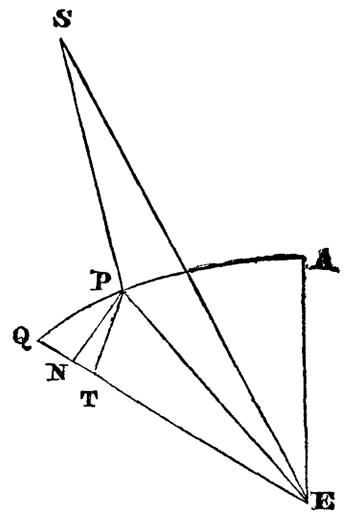
It is entirely by accident that their results are exact in the case of the Moon. Had the periodic time of the moon been different from what it is, observations would have pointed out the error of their conclusions.

The differential equations and the expressions for the forces acting on the moon might have been taken from Laplace or other authors, but it is hoped that the manner in which the differential equations are deduced, may afford an excuse for the introduction of the investigation. The method of limits, the undisputed invention of Newton, seems deserving of more attention than is nowpaid to it.

The differential equations are at once deduced without the intervention of rectangular co-ordinates which in the present case would unnecessarily have added to the length of the process, but in other respects advantage has been taken of the facilities offered by Laplace's investigation of the Lunar motions which may be considered as the most satisfactory and as the most accurate that has appeared.

(I)

Investigation of the differential equations of the orbit described by a body moving in a fixed plane about a fixed centre, when acted on by two forces, one directed to that centre, and the other in a direction perpendicular to the radius vector.



Let APQ be the orbit described about the centre E; let PE $=x$; AEP $=\nu$; velocity at P$=v$; the force at P in direction PN perpendicular to PE$=$P; the centripetal force at P$=$R; area AEP$=z$ and the time of describing PQ $=\Delta t$; the sign Δ expressing a difference.

Then if PT be drawn perpendicular to EQ we may thus proceed to investigate the relation of ν and x

$$\frac{xd\nu}{dx} = \text{limit} \frac{PE \times \Delta}{\Delta PE} = \text{lim.} \frac{PT}{QT} = \text{lim.} \frac{PT}{\sqrt{PQ^2 - PT^2}} = \text{lim.} \frac{1}{\sqrt{\frac{PQ^2}{PT^2} - 1}} \quad (1)$$

$$\text{now } v = \text{lim.} \frac{PQ}{\Delta t} = \frac{dAP}{dt} \text{ or } dAP = vdt$$

$$\text{Therefore lim.} \frac{PQ}{PT} = \text{lim.} \frac{PQ \times QE}{2\,PQE} = \frac{PE}{2} \times \frac{dAP}{dAPE} = \frac{PE}{2} \times \frac{vdt}{dz} \quad (2)$$

To investigate the value of $\frac{dt}{dz}$ we may consider z a function of t, then by Taylor's Theorem

$$\Delta z = \frac{dz}{dt} \Delta t + \frac{d^2 z}{2dt^2} \Delta t^2 + \frac{d^3 z}{6dt^3} \Delta t^3 + \&c. \quad (3)$$

Δz may also be considered as consisting of two parts, one part $\left(= \frac{dz}{dt} \Delta t\right)$ is independent of the two forces R and P, which between P and Q are variable both in quantity and in direction, and the other part arises from the action of these forces during the time Δt. This part we may suppose also produced by an equivalent constant force acting during the time Δt in a direction perpendicular to PE, and we may represent this force by $P + q \Delta t$, q being a variable but determinate quantity, then by the property of uniformly accelerated motion

$$\Delta z = \frac{dz}{dt} \Delta t + \frac{x}{4} (P + q \Delta t) \Delta t^2 \qquad (4)$$

comparing equations (3) and (4)

$$\frac{d^2 z}{2dt^2} + \frac{d^3 z}{6dt^3} \Delta t + \text{\&c.} = \frac{x}{2} (P + q \Delta t)$$

or taking the limits of this equation

$$\frac{d^2 z}{dt^2} = \frac{Px}{2} \text{ or } d^2 z = \frac{Px dt^2}{2} \qquad (5)$$

$$\text{But } \frac{dz}{d\nu} = \text{lim. } \frac{\Delta z}{\Delta \nu} = \frac{x^2}{2} \text{ or } dz = \frac{x^2 d\nu}{2} \qquad (6)$$

Multiplying together equations (5) and (6)

$$dz\, d^2 z = \frac{P}{4} x^3 d\nu\, dt^2$$

and by integration

$$dz^2 = \left(h + \int \frac{Px^3}{2} d\nu\right) dt^2 \qquad (7)$$

h being a constant quantity.

Then from equations (7) (2) and (1) we obtain

$$d\nu = \frac{dx}{x\sqrt{\frac{v^2 x^2}{4} \times \frac{1}{h + \int \frac{1}{2} P x^3 d\nu} - 1}} \qquad (8)$$

also by equations (7) and (6)

$$dt = \frac{x^2 d\nu}{\sqrt{4h + \int 2 P x^3 d\nu}} \qquad (a)$$

But it is necessary to eliminate v from equation 8.

Now $\frac{dv}{dt} = \lim. \frac{\Delta v}{\Delta t}$ = Force in direction of the Tangent

$= P \lim. \frac{PT}{PQ} - R \lim. \frac{QT}{PQ} = P x \frac{d\nu}{dAP} - R \frac{dx}{dAP}$

Therefore because $v = \frac{dAP}{dt}$

$v\,dv = P x\,d\nu - R\,dx$

or by integration

$v^2 = \int 2 P x\,d\nu - \int 2 R\,dx + k$, k being a constant quantity (q)

Hence equation (8) becomes

$$d\nu = \frac{dx}{x^2 \sqrt{\frac{\int P x\,d\nu - \int R\,dx + \frac{1}{2} k}{2h + \int P x^3\,d\nu} - \frac{1}{x^2}}} \qquad (b)$$

This equation of first fluxions is not so convenient for integrating by approximation as the equation of second fluxions which can easily be deduced from it, as follows. Both, however, will be used in the subsequent investigations.

Substituting $u = \frac{1}{x}$ and making $y = \int P u^{-1} d\nu + \int R u^{-2} du + \frac{1}{2} k$

and $w = 2h + \int P u^{-3} d\nu$, we obtain by squaring, &c.

$$d\nu^2 \left(\frac{y}{w} - u^2\right) = du^2 \qquad (9)$$

differencing this equation, making $d\nu$ constant

$$\frac{dy}{2w\,du} - \frac{y\,dw}{2w^2\,du} - u = \frac{d^2u}{d\nu^2} \qquad (10)$$

but by equat. (9)

$$\frac{y\,dw}{w^2\,du} = \frac{y}{w} \times \frac{dw}{w\,du} = \left(\frac{du^2}{d\nu^2} + u^2\right) \frac{dw}{w\,du}$$

therefore by equation (10).

$$\frac{1}{2w}\left(\frac{dy}{du} - \left(\frac{du^2}{d\nu^2} + u^2\right)\frac{dw}{du}\right) - u = \frac{d^2u}{d\nu^2} \qquad (11)$$

$$\text{Now} \frac{dy}{du} = Pu^{-1} \frac{d\nu}{du} + Ru^{-2} \qquad (11)$$

$$\text{and} - \left(\frac{du^2}{d\nu^2} + u^2\right)\frac{dw}{du} = -\frac{Pu^{-3}du}{d\nu} - \frac{Pu^{-1}d\nu}{du}$$

hence we easily deduce from equation (11)

$$\left(\frac{d^2u}{d\nu^2} + u\right)\left(1 + \int \frac{P}{2h} u^{-3} d\nu\right) + \frac{P\,du}{4hu^3d\nu} - \frac{R}{4hu^2} = o \qquad (c)$$

The equation a is equivalent to the first of the equations (L) of Laplace,* and the equation (c) is equivalent to the second of the equations (L) when $s = o$.

(II)

Application to the Lunar Orbit.

1. If we suppose the moon only attracted towards the earth, $P = o$, and R varies inversely as the square of the distance from the earth. Let $R = \frac{M}{x^2}$, M representing the sum of the masses of the moon and earth. Let also $a(1 + e)$ and $a(1 - e)$ represent the greatest and least distances of the moon from the earth on this hypothesis: these distances must be invariable, because the centripetal forces (the force P being $= o$ on this hypothesis) are equal at equal distances on each side of the apsids. We can obtain the values of the constant quantities h and k of the equation (b) in the following manner. Equation (9) becomes $v^2 = -\int \frac{2M\,dx}{x^2} + k = \frac{2M}{x} + k$

therefore V and V' representing the velocities at Perigee and Apogee respectively

$$V^2 = \frac{2M}{a(1-e)} + k;\ V'^2 = \frac{2M}{a(1+e)} + k$$

When $P = o$ equation (7) gives

* Mec. cel. p. 181. tom. 3.

$\frac{d z^2}{d t^2} = h$, therefore at the Apsids

$v^2 = \frac{4 d z^2}{d t^2 \times dist.^2} = \frac{4 h}{dist^2}$

hence $\frac{2 M}{a(1-e)} + k = \frac{4 h}{a^2 (1-e)^2}$

$\frac{2 M}{a(1+e)} + k = \frac{4 h}{a^2 (1+e)^2}$

These two equations give

$h = \frac{1}{4} a (1-e^2) M$

$k = -\frac{M}{a}$

Substituting these values in equation (b) and making $P = o$ and $R = \frac{M}{x^2}$

$$d\nu = \frac{d x}{x^2 \sqrt{\frac{2}{x} - \frac{1}{a} \over a(1-e^2)} - \frac{1}{x^2}} \qquad (15)$$

The integration of this equation (making the longitude of Perigee $= \pi$), gives $\nu = \text{arc}\left(\cos = \frac{a(1+e^2)}{e x} - \frac{1}{e}\right) + \pi$

or $\cos(\nu - \pi) = \frac{a(1-e^2)}{e x} - \frac{1}{e}$

and therefore

$$x = \frac{a(1-e^2)}{1 + e \cos(\nu - \varpi)} \qquad (16)$$

This is the equation of an ellipse, the centre of force being in the focus.

The second law of Kepler follows from this conclusion.

It may be remarked here that the first law of Kepler follows from equation (7) when $P = o$, for it gives $z = h^{\frac{1}{2}} t$, as we may suppose z and t to commence together, and therefore the areas about a fixed centre are proportional to the times of describiug them. Also by help of equation (16) we deduce a conclusion that

will be of use hereafter, and which also proves the third law of Kepler.

Because $t = \frac{z}{h^{\frac{1}{2}}} = \frac{2z}{\sqrt{a(1-e^2)M}}$; consequently,

since by equat. (16) the orbit is an ellipse,

the periodic time is as $\frac{(axis\ maj.)^{\frac{3}{2}}}{\sqrt{M}}$ (17)

and hence about the same centre of force, neglecting the masses of the revolving bodies, the squares of the periodic times are as the cubes of the greater axes.

2. Supposing the moon also acted on by the sun. A small force P exists, and also a small alteration of the force R takes place, and then the integral of the equation (b) is less easily obtained than that of equation (c). Now as the values of P and R depend partly on the relation of u and v, we cannot exactly ascertain the former without knowing the latter; therefore previously to the integration of equat. (c), we must use approximate values of these quantities obtained by the integration of equation (b), when $P = o$ and $R = \frac{M}{x^2}$.

By equation (16)

$$u = \frac{1+e\cos(v-\varpi)}{a(1-e^2)}$$

or if we regard only the first power of the excentricity,

$$u = \tfrac{1}{a}(1+e\cos(v-\pi)) \qquad (18)$$

also $h = \frac{1}{4}aM$

Therefore when $P = o$, equation (a) gives

$$dt = \frac{a^{\frac{3}{2}}dv}{(1+e\cos(v-\varpi))^2 M^{\frac{1}{2}}} = \frac{a^{\frac{3}{2}}dv}{M^{\frac{1}{2}}}(1-2e\cos(v-\pi))$$

By integration,

$$t = \frac{a^{\frac{3}{2}}}{M^{\frac{1}{2}}} (\nu - 2e \sin (\nu - \pi))$$ which requires no correction, as the origin of t is arbitrary.

The above is relative to the moon about the earth; but if we suppose ν' and a' to be the same quantities relative to the earth about the sun, or rather the sun about the earth as ν and α are relative to the moon about the earth.

$$t = \frac{a'^{\frac{3}{2}} \nu'}{S^{\frac{1}{2}}}$$

supposing the earth's orbit without excentricity and S (the mass of the sun) very great in comparison of M.

Hence $\frac{a^{\frac{3}{2}}}{M^{\frac{1}{2}}} (\nu - 2e \sin (\nu - \pi)) = \frac{a'^{\frac{3}{2}} \nu'}{S^{\frac{1}{2}}}$

or making $m = \frac{a^{\frac{3}{2}} S^{\frac{1}{2}}}{a'^{\frac{3}{2}} M^{\frac{1}{2}}} = \frac{\textit{Periodic time moon}}{\textit{Periodic time earth}}$ nearly (19)

$$\nu' = m\,\nu - 2\,em \sin (\nu - \pi)$$

Having thus obtained approximate values of u and ν' in terms of ν, we proceed to find the approximate values of P and R for substitution in the equation (c)

In the preceding figure E, P and S represent the places of the earth, moon and sun.

$$SE = a'; \quad EP = \frac{1}{u} \text{ an } \quad SEP = \nu - \nu'$$

The attraction between S and E will be represented by $\frac{S}{SE^2}$, the mass of the earth being neglected in comparison of that of the sun.

The difference of the forces of S on P and on E in a direction parallel to $SE = S\left(\frac{SE}{SP^3} - \frac{1}{SE^2}\right)$

The effect of this difference in direction EP opposite to $E =$ $S\left(\frac{SE}{SP^3} - \frac{1}{SE^2}\right) \cos(v - v')$

and in direction perpendicular to $PE = S\left(\frac{SE}{SP^3} - \frac{1}{SE^2}\right) \sin(v - v')$

also the force of S in direction $PE = S \times \frac{PE}{PS^3}$

Hence $R = \frac{M}{PE^2} + S\left(\frac{PE}{SP^3}\right) - S\left(\frac{SE}{SP^3} - \frac{1}{SE^2}\right) \cos(v - v')$

and $P = S\left(\frac{SE}{SP^3} - \frac{1}{SE^2}\right) \sin(v - v')$

or in consequence of the substitution $m = \frac{a^{\frac{3}{2}} S^{\frac{1}{2}}}{\acute{a}^{\frac{3}{2}} M^{\frac{1}{2}}}$, if we represent M the sum of the masses of the earth and moon by unity

$$R = \frac{1}{PE^2} + \frac{m^2 \acute{a}^3}{a^3} \left\{\frac{PE}{SP^3} - \left(\frac{SE}{SP^3} - \frac{1}{SE^2}\right) \cos(v - v')\right\}$$

$P = \frac{m^2 \acute{a}^3}{a^3} \left(\frac{SE}{SP^3} - \frac{1}{SE^2}\right) \sin(v - v')$

Now $SP = \sqrt{\acute{a}^2 + \frac{1}{u^2} - \frac{2\acute{a}}{u}(\cos v - v')} = \acute{a}\sqrt{1 - \frac{2\acute{a}u - 1}{u^2 \acute{a}^2}\cos(v - v')}$

hence neglecting quantities of the order $\frac{1}{\acute{a}5}$

$\frac{1}{SP^3} = \frac{1}{\acute{a}^3}\left(1 + \frac{3\cos(v - v')}{u\acute{a}}\right)$

Therefore if we neglect quantities of the order $\frac{m^2}{\acute{a}}$

$R = u^2 + \frac{m^2}{a^3 u} - \frac{3m^2}{a^3 u}\cos^2(v - v') = u^2 - \frac{m^2}{2a^3 u}(1 + 3\cos(2v - 2v'))$

$P = -\frac{3m^2}{a^3 u}\cos(v - v')\sin(v - v') = -\frac{3m^2}{2a^3 u}\sin(2v - 2v')$

now as $m^2 = \left(\frac{\textit{Per. Time Moon}}{\textit{Per. Time Earth}}\right)^2$ nearly $= \frac{1}{178}$ nearly

and a being unity $\frac{1}{\acute{a}} = \frac{\textit{Parallax Sun}}{\textit{Parallax Moon}}$ nearly $= \frac{1}{400}$ nearly; $\frac{1}{a}$ may be

considered of the order m^2. Therefore in the above values of R and P quantities of the order m^4 are neglected.

These values of R and P are next to be substituted in equation (c)

1. $\frac{P}{2hu^3} = -\frac{3m^2}{4ha^3u^4}\sin(2\nu - 2\nu') = -\frac{3m^2a}{4h}\sin(2\nu - 2\nu') \times$
$(1-4\ e\ \cos(\nu-\pi))$ by equat. (18)

But $\sin(2\nu - 2\nu') = \sin(2\nu - 2m\nu + 4em\sin(\nu-\pi))$
$= \sin(2\nu - 2m\nu) + 4em\sin(\nu-\pi)\cos(2\nu - 2m\nu)$
$= \sin(2\nu - 2m\nu) + 2em\sin(3\nu - 2m\nu - \pi) - 2em\sin(\nu - 2m\nu + \pi)$

$$\text{hence } \frac{P}{2hu^3} = -\frac{3m^2a}{4h}\left\{\begin{array}{l}\sin(2\nu - 2m\nu) \\ -(2-2m)\,e\sin(3\nu - 2m\nu - \pi) \\ -(2+2m)\,e\sin(\nu - 2m\nu + \pi)\end{array}\right\}$$

consequently

$$\frac{1}{2h}\int\frac{Pd\nu}{u^3} = \frac{3m^2a}{4h}\left\{\begin{array}{l}\frac{2}{2-2m}\cos(2\nu - 2m\nu) \\ -\frac{2-2m}{3-2m}\,e\cos(3\nu - 2m\nu - \pi) \\ -\frac{2+2m}{1-2m}\,e\cos(\nu - 2m\nu + \pi)\end{array}\right.$$

Without considering the disturbing force of the Sun, we found M being unity that $h = \frac{1}{4}a(1-e^2) = \frac{1}{4}a$ neglecting the second power of the excentricity. If therefore we make $h = \frac{1}{4}a + h'$ h' will be a quantity of the order of the disturbing force or of m^2

Also without the disturbing force of the sun, equation (c) becomes

$$\frac{d^2u}{d\nu^2} + u - \frac{1}{4h} = 0$$

hence $\left(\frac{d^2u}{d\nu^2}+u\right)\int\frac{Pd\nu}{2hu^3}=\frac{3m^2}{a}\left\{\begin{array}{l}\frac{1}{2-2m}\cos(2\nu-2m\nu)\\ -\left(\frac{2-2m}{3+2m}\right)e\cos(3\nu-2m\nu-\pi)\\ -\left(\frac{2+2m}{1-2m}\right)e\cos(\nu\quad 2m\nu+\pi)\end{array}\right.$

neglecting quantities multiplied by h' m^2 as being of the order m^4

2. $\frac{Pdu}{4hu^3d\nu}=\frac{3}{2}m^2e\sin(2\nu-2\acute{\nu})\sin(\nu-\pi)(1-3e\cos(\nu-\pi))$

$=\frac{3m^2}{4a}\left\{\begin{array}{l}e\cos(\nu-2m\nu+\pi)\\ -e\cos(3\nu-2m\nu-\pi)\end{array}\right.$

3. Lastly $-\frac{R}{4hu^2}=-\frac{1}{a+4h'}+\frac{m^2}{2a}-\frac{3m^2}{2a}e\cos(\nu-\pi)$

$+\frac{3m^2}{2a}\left\{\begin{array}{l}\cos 2\nu-2m\nu\\ -\frac{2+4m}{2}e\cos(\nu-2m\nu+\pi)\\ -\frac{3-4m}{2}e\cos(3\nu-2m\nu-\pi)\end{array}\right.$

Substituting these quantities in equation (c) it becomes

$$\frac{du^2}{d\nu^2}+u-\frac{1}{a+4h'}+\frac{m^2}{2a}-\frac{3m^2}{2a}e\cos(\nu-\pi)+\frac{3m^2}{2a}\left(\frac{2-2m}{1-m}\right)\cos(2\nu-2m\nu)$$
$$-\frac{3m^2}{a}\left\{\begin{array}{l}\frac{1+2m}{2}\\ +\frac{2+2m}{1-2m}\end{array}\right\}e\cos(\nu-2m\nu+\pi)-\frac{3m^2}{a}\left\{\begin{array}{l}1-m\\ +\frac{2-2m}{3-2m}\end{array}\right\}e\cos(3\nu-3m\nu)$$
$$=0. \qquad (d)$$

Each of the terms after the four first terms are contained in the form $m^2K\cos(c\nu-\pi)$: it is therefore required to integrate an equation of the form

$$\frac{d^2u}{d\nu^2}+u-G+m^2K(\cos c\nu-\pi)+\&c=0$$

G and K being constant quantities.

(III.)

Integration of the Equation.

$$\frac{d^2u}{dv^2} + u - G + m^2K \cos (cv - \pi) = o$$

G and K being constant quantities, and

1. $c =$ *any number greater or less than unity,*
2. $c = 1$ *and* m^2 *of any magnitude.*
3. $c = 1$ *and* m^2 *a very small quantity.*

The subsequent mode of integration will be more readily understood by a short prefatory explanation.*

Let us suppose we have in any manner arrived at an equation. $d^m \varphi\ (x, y) = o$, when $x = o$ and dx is constant, m representing *any number* not less than n, we can then conclude

$\varphi\ (x, y) = c_1 + c_2 x + c_3 x^2 \ldots\ldots + c_n\ x^{n-1}$

The equation $d^m\ \varphi\,(x, y) = o$ may be called the m^{th} *particular* fluxion of this equation with reference to $x = o$.

If we have only equations of particular fluxions commencing with the order n, the values of c_1, c_2, c_3, &c. are arbitrary. But if there be preceding equations of particular fluxions not contained

* The mode of integration here shortly explained derives its advantage from the method of finding fluxions *per saltum*. Theorems for this purpose were given by me in a paper read before this Academy in the year 1798, and published in the Seventh Volume of the Transactions. These theorems were considerably extended, and applied both to *the direct and inverse reduction of analytical functions* in a work which I had prepared for the press, but having adopted a notation differing both from the fluxional notation, and that of the differential calculus, I was for many years deterred from publishing it. Lately, however, I have again resumed the subject, changing the notation into the usual notation of the differential calculus, and I hope soon to offer the result to the notice of mathematicians. The method of integration here used belongs to that division of the work entited " On the inverse reduction of analytical functions."

in the form $d^m\ \phi\ (x, y) = o$, such equations will serve for determining certain of the values of c_1, c_2 &c.

It will be generally found more convenient in reference to the converse of theorems for finding fluxions *per saltum* to use the fluxional equation.

$$\frac{d^m\ \phi\ (x, y)}{1\ .\ 2\ .\ .\ m} = o$$

which may be denoted by

$\underline{d}^m\ \phi\,(x, y) = o$, the denominator being understood by the line under d and by the order of the fluxion, so that

$$\underline{d}^m\ \phi\ (x, y) = o$$

may with respect to the value of $x = o$ and the denominator be called a *particular divided fluxional equation,* of which the integral is

$$\phi\,(x, y) = c_1 + c_2\,x + c_3\,x^2 \;---\; c_n\ x^{n-1}$$

This mode of proceeding is applicable to the summation of many series, to finding generating functions, to the integration of equations of finite differences and of many fluxional equations, more particularly those equations called linear equations.

To proceed with the equation (e)

It fluxion gives

$$d^3\ u + d\,u\,d\,\nu^2 = m^2\ Kc\ \sin\,(c\nu - \varpi)\ d\ \nu^3 \qquad (1)$$

1. Taking the n—3 *particular fluxion* when $\nu = o$ and $d\ \nu$ is constant, n being even and not less than 4.

$$d^n\ u + d^{n-2}\,u\,d\,\nu^2 = m^2\,(-1)^{\frac{n-4}{2}}\,Kc^{n-2}\,\cos\,\varpi\ d\nu^n \qquad (2)$$

2. Taking the n—3 *particular fluxion* when $\nu = o$ and $d\,\nu$ is con-

stant, n being odd and not less than 3.

$$d^n u + d^{n-2} u \, d\nu^2 = -m^2 (-1)^{\frac{n-3}{2}} K c^{n-2} \sin \varpi \, d\nu^n \qquad (3)$$

Let z be a function of u

such that when $\nu = o$

$d^n u = \underline{d}^n z$

Then equations (2) and (3) become

$$\underline{d}^n z + \underline{d}^{n-2} z \, d\nu^2 = (-1)^{\frac{n-2}{2}} m^2 K c^{n-2} \cos \varpi \, d\nu^n \qquad (4)$$

$$\underline{d}^n z + \underline{d}^{n-2} z \, d\nu^2 = -(-1)^{\frac{n-3}{2}} m^2 K c^{n-2} \sin \varpi \, d\nu^n \qquad (5)$$

Now when $\nu = o$

the nth *particular divided fluxion* of $\frac{\nu^3}{1+c^2\nu^2}$

when n is odd and not less than $3 = d\nu^3 \, \underline{d}^{n-3} \frac{1}{1+c^2\nu^2}$

$= (-1)^{\frac{n-3}{2}} c^{n-3} d\nu^n$ and when n is even $= o$

and the nth *particular divided fluxion of*

$\frac{\nu^2}{1+c^2\nu^2} \begin{cases} \text{when } n \text{ is even and not less than } 2 = (-1)^{\frac{n-2}{2}} c^{n-2} d\nu^n \\ \text{when } n \text{ is odd } = o \end{cases}$

Hence equations (4) and (5) are deduced from the integral

$$(1+\nu^2) z = c_1 + c_2 \nu + c_3 \nu^2 - \frac{m^2 K c \sin \varpi . \nu^3 + m^2 K \cos \varpi . \nu^2}{1+c^2\nu^2} \qquad \text{or}$$

$$z = \frac{c_1 + c_2\nu + c_3\nu^2}{1+\nu^2} - \frac{m^2 K (c\nu^3 \sin \varpi + \nu^2 \cos \varpi)}{(1+c^2\nu^2)(1+\nu^2)} \; (w) \qquad (6)$$

Where c_1, c_2, c_3 are constant quantities, of which two are arbitrary with respect to the given equation.

The former fraction $= \frac{c_1 - c_3 + c_2 \nu}{1+\nu^2} - c_3$ and therefore may be written

$$\frac{c_1 + c_2 \nu}{1 + \nu^2} - c_3$$

The fraction (w) may be readily resolved into two fractions of the form

$$\frac{p+q\nu}{1+c^2\nu^2} + \frac{r+s\nu}{1+\nu^2}$$

where $p + r = o \qquad q + s\,c^2 = -m^2\,Kc \sin \pi$

$q + s = o \qquad p + r\,c^2 = -m^2\,K \cos \pi$

$$p = -\frac{m^2\ K \cos \varpi}{1-c^2},\ q = -\frac{m^2\ Kc \sin \varpi}{1-c^2}$$

$$r = \frac{m^2 K \cos \varpi}{1-c^2},\ s = \frac{m^2\ Kc \sin \varpi}{1-c^2}$$

the fraction $\frac{r+s\nu}{1+\nu^2}$ may be considered as contained in

$\frac{c_1+c_2\nu}{1+\nu^2}$, c_1 and c_2 being arbitrary.

whence $z = \frac{c_1+c_2\nu}{1+\nu^2} + \frac{p+q\nu}{1+c^2\nu^2} - c_3$

Therefore, when n is odd and $\nu = o$

$$d^n u = \underline{d}^n z = (-1)^{\frac{n-1}{2}} c_2\, d\nu^n + (-1)^{\frac{n-1}{2}} q\, c^{n-1}\, d\nu^n$$

when n is even and $\nu = o$

$$d^n u = \underline{d}^n z = (-1)^{\frac{n}{2}} c_1\, d\nu^n + (-1)^{\frac{n}{2}} p\, c^n\, d\nu^n$$

These equations are the nth ***particular fluxions*** of

$u = c_1 \cos \nu + c_2 \sin \nu + p \cos c\,\nu + \frac{q}{c} \sin c\,\nu - c_3$

comparing the second fluxion of this equation with the given equation

$$-\overset{\cdot\cdot}{c_3} = G$$

Therefore substituting for p and q their values above given

$$u = c_1 \cos \nu + c_2 \sin \nu - \frac{m^2\ K}{1-c^2} \cos (c\,\nu - \pi) + G$$

Case 2. When $c = 1$

Equation (6) of this article gives

$$z = \frac{c_1 + c_2 \nu}{1+\nu^2} - c_3 - \frac{m^2 K(\nu^3 \sin \varpi + \nu^2 \cos \varpi)}{(1+\nu^2)^2} \quad (v)$$

let the term v be represented by

$$\frac{p\nu^2 + q\nu^3}{(1+\nu^2)^2}$$

let also the part of u corresponding to v be called w.

1. n being even and $\nu = o$

$$\underline{d}^n v = (-1)^{\frac{n-2}{2}} \left(\tfrac{n}{2}\right) p\, d\nu^n$$

2. n being odd and $\nu = o$

$$\underline{d}^n v = (-1)^{\frac{n-3}{2}} \left(\tfrac{n-1}{2}\right) q\, d\nu^n. \quad \text{Therefore } \underline{d}^n w = (-1)^{\frac{n-2}{2}} \left(\tfrac{p}{2}\right) \underline{d}\, \nu^{n-1} d\nu$$

$$\text{or } \underline{d}^n w = -(-1)^{\frac{n-1}{2}} \left(\tfrac{q}{2} d\nu^{n-1} d\nu - \tfrac{q}{2} d\nu^n\right)$$

according as n is even or odd.

These two equations are the *particular divided fluxions of*

$$w = -\frac{p}{2}\nu \sin \nu + \frac{q}{2}\nu \cos \nu - \frac{q}{2} \sin \nu$$

the latter term may be neglected, being contained in $c_2 \sin \nu$. Hence substituting for p and q

$$u = c_1 \cos \nu + c_2 \sin \nu - \tfrac{1}{2} m^2 K\nu \sin (\nu - \varpi) + G$$

Case 3. When $c = 1$ and m^2 is a very small quantity, the following integration will be exact to the first power of m^2.

By the same substitution as in the preceding case

$$z = \frac{c_1 + c_2\nu}{1+\nu^2} - \frac{p\nu^2 + q\nu^3}{(1+\nu^2)^2} - c_3$$

$$\text{now } \frac{c_1}{1+\nu^2} - \frac{p\nu^2}{(1+\nu^2)^2} = c_1\left(\frac{1}{1+\nu^2} - \frac{p\nu^2}{c_1(1+\nu^2)^2}\right) = \frac{c_1}{1+\nu^2} \times \frac{1}{1 + \frac{p\nu^2}{c_1(1+\nu^2)}}$$

$$\text{nearly} = \frac{c_1}{1 + \left(1 + \frac{p}{c_1}\right)\nu^2}$$

$$\text{Hence } z = \frac{c_1}{1 + \left(1 + \frac{p}{c_1}\right)\nu^2} + \frac{c_2 \nu}{1 + \left(1 + \frac{q}{c_2}\right)\nu^2} \quad \text{nearly}$$

Then $d^n u = d^n z = c_1 (-1)^{\frac{n}{2}} \left(1 + \frac{p}{c_1}\right)^{\frac{n}{2}} d\nu^n$, n even and $\nu = o$

$d^n u = d^n z = c_2 (-1)^{\frac{n-2}{2}} \left(1 + \frac{q}{c_2}\right)^{\frac{n-1}{2}} d\nu^n$, n odd and $\nu = o$

These two equations are the *particular fluxions of*

$u = c_1 \cos \left(1 + \frac{p}{c_1}\right)^{\frac{1}{2}} \nu + c_2 \sin \left(1 + \frac{q}{c_2}\right)^{\frac{1}{2}} \nu + G$

where $p = m^2 K \cos \pi$ and $q = m^2 K \sin \pi$

A first approximation therefore gives the value of u in periodical terms, instead of in terms without the periodical signs as occur in the complete solution of the differential equation obtained in case the 2d.

(IV.)

Further application to the Lunar Orbit.

It was found by Equation (18) Art. (II.) that without the disturbing force of the Sun

$u = \frac{1}{a}\left(1 + e \cos (\nu - \pi)\right) = \frac{1}{a} + \frac{e}{a} \cos \nu \cos \pi + \frac{e}{a} \sin \nu \sin \pi$, regarding only the first power of the excentricity.

Also without the disturbing force of the Sun by the preceding article the integral of equation (d) of article (II.) becomes

$$u = c_1 \cos \nu + c_2 \sin \nu + \frac{1}{a}$$

comparing these values of u

$$c_1 = \frac{e}{a} \cos \pi \qquad c_2 = \frac{e}{a} \sin \pi$$

Hence in case 3 of the preceding article, because

$K = -\frac{3e}{2a}$ and therefore $p = -\frac{3m^2 e \cos \pi}{2a}$, $q = -\frac{3m^2 e \sin \pi}{2a}$

$$c_1 \cos\left(1+\frac{p}{c_1}\right)^{\frac{1}{2}} v + c_2 \sin\left(1-\frac{q}{c_2}\right)^{\frac{1}{2}} v$$
$$= \frac{e}{a}\cos\pi\ \cos\left(1-\frac{3m^2}{2}\right)^{\frac{1}{2}} v + \frac{e}{a}\sin\pi\sin\left(1-\frac{3m^2}{2}\right)^{\frac{1}{2}} v =$$
$$= \frac{e}{a}\cos\left(\left(1-\frac{3m^2}{2}\right)^{\frac{1}{2}} v - \pi\right)$$

Consequently by the application of the first and third cases of the preceding article, the integration of equation (d) gives $u = \frac{1}{a+4h^2} - \frac{m^2}{2a} + \frac{e}{a}\cos\left(\left(1-\frac{3m^2}{4}\right)v - \pi\right) + \frac{A^{(0)}}{a}\cos(2v - 2mv)$
$+ \frac{Ae}{a}\cos(v - 2mv + \pi) + \frac{A^{(2)}}{a} e\cos(3v - 2mv - \pi)$ $\qquad (f)$

where $A^{(0)} = -\frac{3m^2}{2}\left(\frac{2-m}{1-m}\right)\frac{1}{(1-(2-2m)^2}$ and therefore is of the order of m^2

$$A = 3m^2\begin{Bmatrix}\frac{1+2m}{2}\\ +\frac{2+2m}{1-2m}\end{Bmatrix}\frac{1}{1-(1-2m)^2}$$ and therefore on account of the divisor $1-(1-2m)^2 = 4m - 4m^2$ is of the order of m

$$A^{(2)} = 3m^2\begin{Bmatrix}1-m\\ +\frac{2-2m}{3-2m}\end{Bmatrix}\frac{1}{1-(3-3m)^2}$$ and therefore is of the order of m^2

Hence $u = \frac{1}{a} + \frac{e}{a}\cos(v - \pi) + \frac{Ae}{a}\cos(v - mv + \pi)$ will be a new value of u, exact to the order of m.

It is clear by the third term of the value of u given by equation (f) that when $\left(1-\frac{3m^2}{4}\right)v - \pi = o$ or a multiple of the circumference, the moon is at perigee, not regarding the periodic terms depending on the place of the sun, and therefore the mean motion of the moon : mean motion of its perigee : : $1 : \frac{3m^2}{4}$ The pro-

gressive motion of the perigee thus found is, as is well known, only about half its real motion.

We must next investigate the effect of the new substitution $u = \frac{1}{a} + \frac{e}{a} \cos(\nu - \pi) + \frac{A e}{a} \cos(\nu - m\nu + \pi)$ on the terms of the equation (c) of art. (1.) and thence deduce a new equation (d) the integration of which will give a new value of u. For this purpose it will only be necessary to compute the variation of equation (d) arising from a variation $\delta u = \frac{A e}{a} \cos(\nu - m\nu + \pi)$ and as the object of our enquiry is the mean motion of the Lunar perigee, it will be only necessary to compute the new terms of the equation (d) of the form $B \cos(\nu - \pi)$, for from a similar term, viz. $-\frac{3m^2 e}{2a} \cos(\nu - \pi)$ arose the term $\cos\left((1 - \frac{3m^2}{4})\nu - \pi\right)$. The integration given in case 3 of the preceding article will then at once give the coefficient from which the mean motion of the perigee is deduced, as far as the new approximation to the value of u is concerned.

To compute the variation of the equation (d) as to the above-mentioned terms.

$$1.\quad \delta - \frac{R}{4hu^2} = -\frac{3m^2 \delta u}{2a^3 hu^4} - \frac{9m^2 \delta u}{2a^3 hu^4} \cos(2\nu - 2\nu') + \frac{3m^2 \delta \nu'}{a^3 h u^3} \sin(2\nu - 2\nu')$$

$$\nu' = m\nu - 2em \sin(\nu - \pi)$$

This gives $\delta \nu' = -2em \sin(\nu - \pi)$, supposing $\delta u = \frac{e}{a} \cos(\nu - \pi)$

Hence nearly $\delta \nu' = -2Aem \sin(\nu - m\nu + \pi)$

when $\delta u = \frac{A}{e} \cos(\nu - 2m\nu + \pi)$

Substituting these values of δu and $\delta \nu$,

The first term or $-\frac{3m^2\,\delta u}{2a^3\,hu^4}$ contains no term of the form $B\cos(\nu-\pi)$

The second term or $-\frac{9m^2\delta u}{2a^3hu^4}\cos(2\nu-2m\nu)$ gives a term

$$-\frac{9m^2eA}{4a^4hu^4}=-\frac{9m^2Ae}{4a}\cos(\nu-\pi) \qquad (1)$$

The third term gives a term

$$-\frac{3m^2eA}{ha^4u^3}\cos(\nu-\pi)=-\frac{3m^2Ae}{a}\cos(\nu-\pi) \qquad (2)$$

2. $\delta\,\frac{Pdu}{4hu^2d\nu}=-\frac{3}{4}\,\delta\,\frac{m^2}{a^4u^4}\frac{du}{d\nu}\sin(2\nu-2\nu')$

$=\frac{6m^2\delta u}{a^4u^5}\cdot\frac{du}{d\nu}\sin(2\nu-2m\nu)-\frac{3m^2}{2a^4u^4}\cdot\frac{d\delta u}{d\nu}\sin(2\nu-2m\nu)$

$+\frac{3m^2du}{a^4u^4d\nu}\,\delta\nu'\cos(2\nu-2m\nu)$

The first and third terms do not afford any term of the required form $B\cos(\nu-\pi)$

but $-\frac{3m^2}{2a^4u^4}\cdot\frac{d\delta u}{d\nu}\sin(2\nu-2m\nu)$ gives

$$\frac{3}{4}(1-2m)\frac{m^2A}{a}e\cos(\nu-\pi) \qquad (3)$$

3. The variation of

$\left\{\frac{d^2u}{d\nu^2}+u\right\}\int\frac{Pd\nu}{2hu^3}=-\delta\left\{\frac{d^2u}{d\nu^2}+u\right\}\int\frac{3m^2}{a^4u^4}d\nu\sin(2\nu-2\nu')$

$=$ (substituting $\frac{1}{a}$ instead of $\frac{d^2u}{d\nu^2}+u$) $\int\frac{12m^2\delta u}{a^5u^5}d\nu\sin(2\nu-2\nu')$

$+\int\frac{6m^2}{a^4u^4}d\nu\,\delta\nu'\cos(2\nu-2\nu')-\left(\frac{dd\delta u}{d\nu^2}+\delta u\right)\int\frac{3m^2}{a^4u^4}d\nu\sin(2\nu-2\nu')$

The first two terms give a term

$$-\frac{6m^2}{a}(1+m)Ae\cos(\nu-\pi) \qquad (4)$$

The third term gives a term

$$-\frac{3m^2}{a}\cdot\frac{(1-2m)^2-1}{4(1-m)}\cos(\nu-\pi) \qquad (5)$$

Hence collecting the terms (1) (2) (3) (4) (5) the co-efficient of the term e cos $(\nu-\pi)$ in the variation of the equation (d) obtained by $\delta u = \frac{A}{e}$ cos $(\nu-2m\nu+\pi)$ will be found $=$

$$-\frac{3m^2}{2a}(5+5m)\,A\,e\cos(\nu-\pi)$$

consequently the coefficient of e cos $(\nu-\pi)$ in the differential equation arising from the substitution of

$$u = \frac{1}{a} + \frac{e}{a}\cos(\nu-\pi) + \frac{Ae}{a}\cos(\nu-m\nu+\pi)$$

is thus found $= -\frac{3m^2}{2a}\left(1 + (5+5m)\,A\right)$

Hence by case 3 Art. (III) the corresponding term in the integral or value of $u = e\cos\left(\left(1-\frac{3m^2}{4}(1+(5+5m)\,A\right)\nu-\pi\right)$

Therefore the mean motion of the perigee is

$\frac{3}{4}m^2\left(1+(5+5m)\,A\right)$ that of the Moon being unity.

$A = \frac{3}{4}\left(\frac{m}{1-m}\right)\left(\frac{1+2m}{2}+\frac{2+2m}{1-2m}\right) = \frac{3}{8}m\,(5+19m)$ nearly;

now $m = \frac{\text{Periodic time moon}}{\text{Periodic time earth}}$ nearly $= 0{,}0748$.

Therefore $A =$, 1801 nearly.

Hence the mean motion of the perigee thus found by the second approximation $=$, 00826 or $2^{\circ}.58',4$ each revolution, the observed mean motion $=$, 00845 $=$ 3. 2, 5 each revolution: the difference is about $\frac{1}{40}$th part of the whole, whereas the first approximation was only about one half of the whole.

It is evident, that if we had also found the variations arising from the other terms of u besides Ae cos $(\nu-m\nu+\pi)$, the results would, (as will easily appear by considering the values of these quantities in comparison of Ae cos $(\nu-m\nu+\pi)$ and the nature of the combinations required to form the angle $(\nu-\pi)$ have had only a small effect on the mean motion of the perigee.

The above, it is conceived, is sufficient for two purposes.

1. To shew by a compendious process that the mean motion of the Lunar perigee may be satisfactorily accounted for by the Newtonian Theory of Gravitation.

2. To shew that the term e cos $(\nu - \varpi)$ occurring in the first approximation for the value of u becomes by succeeding approximations of the form e cos $(c\nu - \pi)$ and therefore if we use this form in the first approximation and then add other terms of the proper combination of angles with indeterminate coefficients, as Laplace has done, considerable advantage will be thus afforded of obtaining a sufficiently exact value of u, for the purpose of an exact determination of the Lunar orbit. No farther difficulty afterwards occurs except that arising from the great length of the computations.

It is remarkable that in the above expressions for the mean motion of the perigee, the part $\frac{3}{4}\, m^2$ arising from the first approximation is nearly equal to $\frac{3}{4}\, m^2\ (5 + 5m)\ A$; because $m =\ ,0748$ and $A =,\ 1814$ and therefore $(5 + 5m)\ A = 0,97$ or unity nearly. But this coincidence is merely accidental, had the periodic time of the moon been double what it is, the first approximation would not have given $\frac{1}{3}$ of the whole motion, but had the Moon revolved about the Earth in one day, the Newtonian result, according to the principles of the 9th sect. of the first book of the Principia, would have been within $\frac{1}{40}$ part of the whole, instead of being within one half only.

Hence we see the results of the investigations of Machin, Walmsly, Mathew Stewart and Frisi are all quite erroneous, as their methods give the mean motion of the perigee $= \frac{3\;{}^{2}}{2}$ nearly, which only affords an exact result when the periodic time of the Moon $= \frac{1}{13}$ that of the Earth nearly.

The common error into which they have fallen is, as appears to me, pointed out in a paper of mine in the 8th volume of the transactions of this academy.

Although the above investigation is only exact to the first power of the excentricity of the lunar orbit, yet it appears that the excentricity does not enter the above expression for the motion of the perigee, and therefore it follows that the excentricity of the lunar orbit has but little effect in the motion of the perigee. In fact only the second powers of the excentricities of the solar and lunar orbits and the second power of the tangent of the inclination of the lunar orbit to the plane of the ecliptic enter into the more exact expression of the mean motion of the lunar perigee.